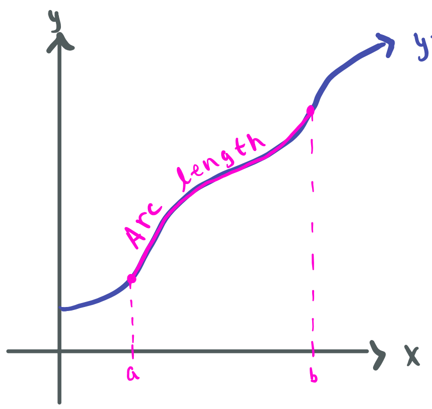
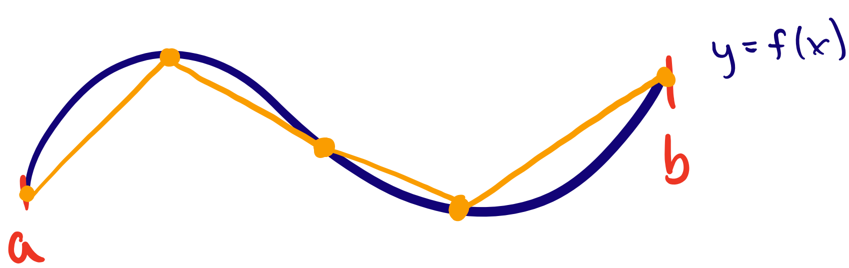
Arc length

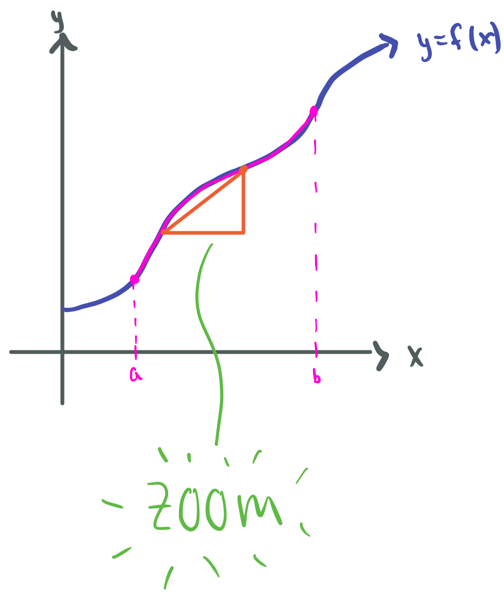
Finding the exact length of a curve using integration techniques
Think of it like laying a string directly on top of a curve, we want to find the length of that string

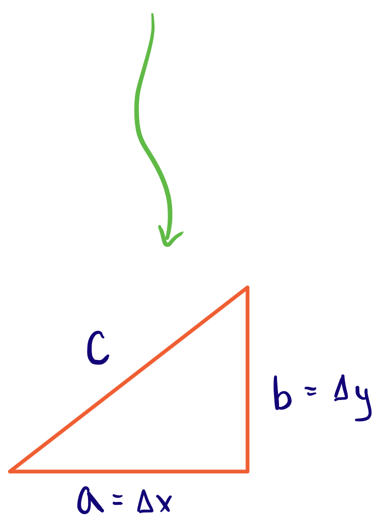
Consider the curve $y = f(x)$



We can approximate the length like so:







We want to find the length of c

$$c = \sqrt{a^2 + b^2} = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

$$= \sqrt{(\Delta x)^2 \left(1 + \frac{(\Delta y)^2}{(\Delta x)^2}\right)} = \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}$$

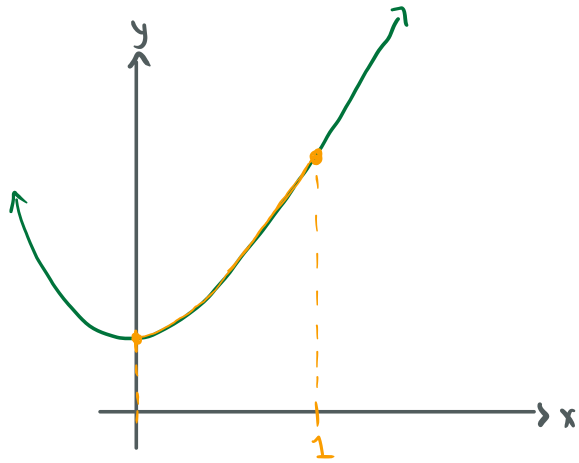
Finally, we get $C = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\, \Delta x$ for the length of a single arbitrary segment of this arc

To obtain the entire Arc length we need to add up an infinite amount of infinitesimal segments ... which we should all know is the power of the Integral. Thus,

$$\boxed{\text{Arc length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx}$$

Ex | Find the exact length of the curve

$y = x^2 + 1$, for $0 \le x \le 1$



$y'(x) = 2x$

$$= \int_0^1 \sqrt{1 + (2x)^2}\, dx$$

$$\boxed{= 1.47 \text{ units}}$$

Ex | Find the exact length of the curve

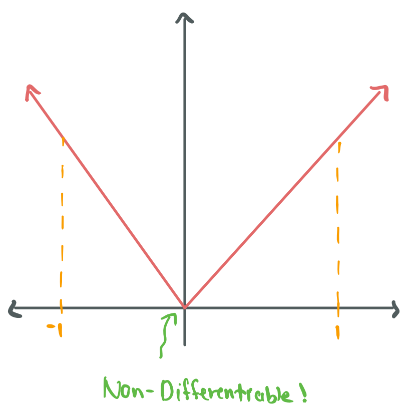
$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1$, for $0 \le x \le 1$

$y'(x) = 2\sqrt{2x}$ $\Rightarrow$ $\int_0^1 \sqrt{1+(2\sqrt{2x})^2}\,dx = \int_0^1 \sqrt{1+8x}\,dx$

$= \boxed{\frac{13}{6} \text{ units}}$

ex) Find the exact length of the Curve

$y = |x|$, for $-1 \le x \le 1$



$$|x| = \begin{cases} -x, & x<0 \\ x, & x>0 \end{cases}$$

$$\int_{-1}^{0} \sqrt{1+(-1)^2}\,dx + \int_0^1 \sqrt{1+(1)^2}\,dx$$

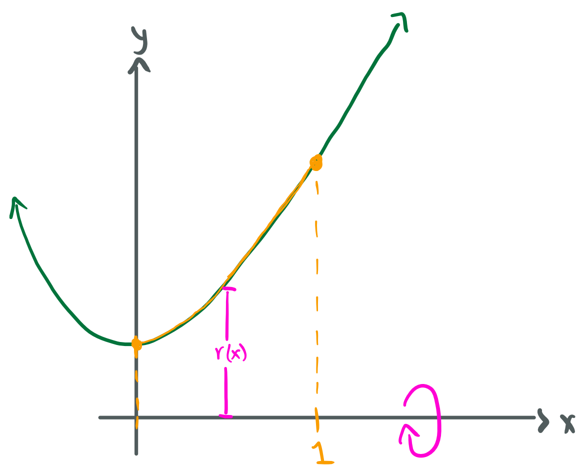
$$= \Big[\sqrt{2}\,x\Big]_{-1}^{0} + \Big[\sqrt{2}\,x\Big]_0^1 = \boxed{2\sqrt{2}}$$

Surface Area

This will not be on your Test or AP Exam

$$SA = 2\pi \int_a^b r \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Ex] Find the Surface Area generated when the Curve $y = x^2 + 1$, for $0 \le x \le 1$ is revolved around the x-axis.

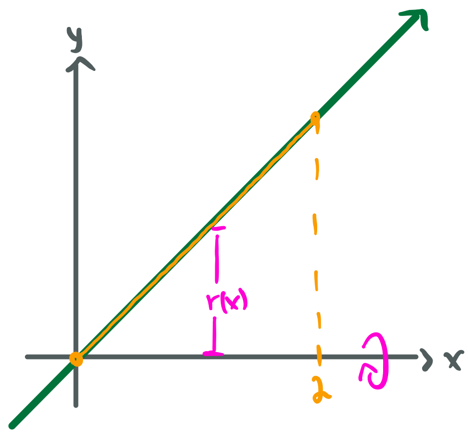


$r = y(x) = x^2 + 1$

$y'(x) = 2x$

$$\Rightarrow 2\pi \int_0^1 (x^2+1)\sqrt{1 + (2x)^2}\, dx = \boxed{13.01 \text{ units}}$$

Ex| Find the Surface Area generated when the Curve $y = x$, for $0 \le x \le 2$ Is revolved around the x-axis.



$r = y(x) = x$

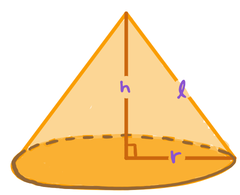
$y'(x) = 1$

$$\Rightarrow 2\pi \int_0^2 x\sqrt{1+1^2}\,dx = 2\pi\sqrt{2}\int_0^2 x\,dx$$

$$= 2\pi\sqrt{2}\left[\tfrac{1}{2}x^2\right]_0^2 = \boxed{17.7 \text{ units}}$$

The equation for SA of a Cone is

$$A_s = \pi r\left(r + \sqrt{h^2 + r^2}\right)$$



Find the SA using this formula & Compare Answers

$$A_s = \pi 2\left(2 + \sqrt{2^2 + 2^2}\right)$$

$$= 2\pi\left(2 + 2\sqrt{2}\right) = \boxed{30.33 \text{ units}}$$

They are different because the circular base isn't accounted for in the initial question!